Problem Set 5 due October 14, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Consider four non-zero vectors $\boldsymbol{c}$, $\boldsymbol{n}$, $\boldsymbol{r}$, $\boldsymbol{l}$ in $\mathbb{R}^2$. What are the conditions these four vectors need to satisfy, in order for:

- $\boldsymbol{c}$ to span the column space of A
- $\boldsymbol{n}$ to span the nullspace space of A
- $\boldsymbol{r}$ to span the row space of A
- $\boldsymbol{l}$ to span the left nullspace of A

for some 2×2 matrix A. If these conditions are satisfied, write down such a matrix A. (keep the given vectors $\boldsymbol{c}$, $\boldsymbol{n}$, $\boldsymbol{r}$, $\boldsymbol{l}$ abstract, i.e. don't just plug in numbers). *(20 points)*

Problem 2: Consider the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} -5 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Invent an algorithm (explain all the steps of the algorithm in words, and explain why it works) which takes general vectors $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ as inputs, and decides whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$. *(15 points)*

Problem 3: Consider the line L spanned by $\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ and the plane $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x + 3y + 6z = 0 \right\}$.

1. Compute any two basis vectors of the plane V. *(5 points)*

2. Compute the projection matrices P_L onto L and P_V onto V. *(10 points)*

3. Compute $P_L + P_V$. The answer should be a very nice matrix. Explain geometrically why you get this answer (hint: it has to do with the relationship between L and V). *(10 points)*

Problem 4: Consider the following lines L_1 and L_2 in 3-dimensional space:

$$L_1 = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} \text{ for } x \in \mathbb{R} \right\} \qquad \text{and} \qquad L_2 = \left\{ \begin{bmatrix} y \\ 2y-1 \\ 3y \end{bmatrix} \text{ for } y \in \mathbb{R} \right\}$$

1. Which of these is a subspace and which is not? *(5 points)*

2. Use least squares to compute the smallest possible distance from a point on the line which is not a subspace to the line which is a subspace. *(5 points)*

3. By minimizing the quantity in part (2), find the points $P \in L_1$ and $Q \in L_2$ for which the distance $|PQ|$ is minimal among all possible choices of a point on either line. *(5 points)*

4. What can you say about the line PQ in relation to the lines L_1 and L_2? *(5 points)*

Problem 5: The equation of a parabola in the plane is $y = ax^2 + bx + c$.

1. Compute a, b, c such that the parabola passes through the points $(1, 0)$, $(2, 4)$, $(-1, -2)$ (don't just guess, use linear algebra to solve for a, b, c). *(10 points)*

2. Compute a, b, c such that the parabola is the best fit for the points $(1, 0)$, $(2, 4)$, $(-1, -2)$, $(-2, 5)$: this means that the sum of the squares of the vertical distances between the parabola and the four given points should be minimum (Hint: this is done similarly to the example of fitting a line, that we did at the end of Lecture 13). *(10 points)*